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SOME REMARKS ON SHALLOW WATER THEORY

by

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INTRODUCTION

The problem of calculating the reflection and transmission effects of floating bodies on surface waves has been treated rather generally by F. John [1]. The object of this report is to derive the approximate theory from the exact hydrodynamical theory for gravity waves of small surface amplitude by making the simplifying assumption that the wave length $\underline{\lambda}$ in the water is sufficiently large compared with the depth $\underline{h}$ of the water. The derivation of this approximate theory for different boundary conditions is based on a paper by J. J. Stoker [2].

In Section 1, Part I, the shallow water theory taking into account the higher order terms will be derived. This theory will then be specialized for four different cases to be treated here in detail.

In Section 2, Part I, it is shown that the approximate theory has to make not only the restriction that the ratio λ/h be small but also that the ratio a/λ be large enough ($a/\lambda > 1$) where 2a is the length of the obstacle.

In Section 1, Part II, the potential function $\underline{\Phi}$ for the exact linear theory with certain initial conditions is derived (see J. J. Stoker [3]) and for this same case the potential function $\underline{\Phi}$ for the approximate theory is derived and the two are compared.

The results obtained in the above-mentioned cases give some general ideas for treating the reflection of waves from an obstacle of finite length in the shallow water theory. The results obtained for different boundary conditions show under what conditions the best result for different parameters can be expected. The comparison of the potential function $\underline{\Phi}_p$ of the exact linear theory to the potential function $\underline{\Phi}_{sh}$ of the approximate linear theory shows that for small λ/h the two solutions differ very little, but one has to be careful in making any conclusions for the amplitudes of reflected waves in the approximate theory. The results in Part I, Section 2, show that,

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In section I, part [illegible] smallness [illegible] energy take into account [illegible] will be derived. This theory will [illegible] generalized [illegible] different cases to be treated here in detail.

In section [illegible] part [illegible], it is shown that [illegible] approximate theory [illegible] only [illegible] that the ratio λ [illegible] be small [illegible] the [illegible] the quantity h/ℓ [illegible] where 2ℓ is the length of the [illegible].

In section [illegible] part [illegible], the potential function Φ for the exact linear theory with certain initial conditions is derived ([illegible] J. J. Stoker [3]) [illegible] this same [illegible] the potential function Φ [illegible] approximate theory [illegible] and the [illegible] compared.

The results obtained in [illegible] three [illegible] mentioned cases give some general ideas [illegible] waves [illegible] finite length in the [illegible] theory. The results [illegible] obtained for different [illegible] under [illegible] conditions [illegible] the best results for different parameters can be [illegible] The comparison [illegible] Φ_p of the exact linear theory [illegible] the function φ_{ap} of the approximate [illegible] the [illegible] in making [illegible] in the [illegible].

at least in a particular case, the ratio a/λ has to be taken into account. We showed one case in order to draw attention to the fact that the parameter a/λ has to be considered. It will be worthwhile to take into account even the higher order terms in h/λ (fifth and sixth order) and to see how much influence this will have on the results obtained here. It should have been done also for different boundary conditions. In principle the taking into account of the higher order terms of h/λ for different boundary conditions doesn't cause any mathematical difficulty, but it is very tedious. Consequently we are calling to attention only some of the facts which we proved, showing also the way of setting up such problems and the rest is left to the people who are interested in the specific cases.

NOTATION

The following table comprises a list of the principal notations employed.

Φ, ϕ	=	potential function
B	=	amplitude of the incident wave
B'	=	amplitude of the reflected wave
T	=	amplitude of the transmitted wave
g	=	gravitational constant
h	=	depth of the water in the direction of y-axis
p	=	pressure
η, w	=	elevation
ρ	=	density of the water
$2a$	=	length of the obstacle
σ	=	frequency of the motion
t	=	time
λ	=	wave length
x, y	=	coordinate axes

PART I

§1.

In the derivation of the approximate theory we start from the potential equation for Φ and integrate it with respect to y from the bottom to the equilibrium position of the top surface (see J. J. Stoker [2], page 29, and J. J. Stoker [4]),

$$(1) \quad \int_{-h}^{0} \Phi_{yy}\, dy = \Phi_y = -\frac{\partial}{\partial x}\left[h\,\Phi_x - \int_{-h}^{0} (y+h)\Phi_{xy} dy\right] = -h\,\Phi_{xx} + \frac{\partial I}{\partial x}$$

where

$$(2) \quad I = \int_{-h}^{0} (h+y)\Phi_{xy} dy = \frac{h^2}{2}\Phi_{xy} - \int_{-h}^{0} \frac{h^2}{2}\Phi_{xyy} dy$$

$$= \frac{h^2}{2}\Phi_{xy} - \frac{h^3}{6}\Phi_{xyy} + \int \cdots$$

(integrating by parts and taking $y = 0$) .

Combining equations (1) and (2) we get:

$$(3) \quad \Phi_y = -h\,\Phi_{xx} + \frac{h^2}{2}\Phi_{xxy} - \frac{h^3}{6}\Phi_{xxyy} + \int \cdots \quad .$$

The conditions to be satisfied by Φ are (for derivation see, for example, Lamb [5] or Milne-Thompson [6])

$$(4) \quad \Phi_{xx} + \Phi_{yy} = 0 \qquad \text{for} \qquad 0 \geqq y \geqq -h$$

$$(5) \quad \Phi_{tt} = -g\,\Phi_y \qquad \text{for} \qquad y = 0$$

$$(6) \quad \Phi_y = -h\,\Phi_x \qquad \text{for} \qquad y = -h \quad .$$

The surface elevation η is given by:

$$(7) \quad \eta(x,t) = \frac{1}{g}\Phi_t \qquad \text{at} \qquad y = 0 \quad .$$

From equations (7) and (5) we obtain

(8) $\eta_t = \frac{1}{g} \Phi_{tt} = - \Phi_y$ at $y = 0$

and so we can write for equation (3)

(9) $- \eta_t = -h \Phi_{xx} + \frac{h^2}{2} \Phi_{xxy} - \frac{h^3}{6} \Phi_{xxyy} + \int \cdots$

or

(10) $- \eta_t = -h \Phi_{xx} + \frac{h^2}{2} \Phi_{xxy} - \frac{h^3}{6} \Phi_{xxyy} + O(h^4)$.

(We assume that Φ_{xxxyy} and Φ_{xxyy} are bounded for all x and t, and for $-h \leqq y \leqq 0$.) The approximate theory called the shallow water theory neglects the higher order terms in h. Taking the lowest order term in h we have

(11) $\eta_t = -h \Phi_{xx}$.

Setting $\eta = w(x)e^{i\sigma t}$ and $\Phi = \phi(x)e^{i\sigma t}$, we obtain

(12) $i\sigma w = -h \phi_{xx}$

or

(13) $\phi = - \frac{i\sigma w}{2h} x^2 + \gamma x + \delta$ (integrating ϕ_{xx} twice) .

From equations (5) and (3) and taking only the lowest order term in h, we obtain

(14) $\Phi_{tt} = gh \Phi_{xx}$

or $-\sigma^2 = gh\phi_{xx}$. Setting $\phi = \bar{\phi}\, e^{imx}$ we get

(15) $-\sigma^2 = -gh\, m^2$

or

(15a) $\sigma = \sqrt{gh}\, m$.

From equations (5) and (3) neglecting the terms of $O(h^4)$ and higher we get

(16) $\Phi_{tt} = gh \Phi_{xx} - \frac{gh^2}{2} \Phi_{xxy} + \frac{gh^3}{6} \Phi_{xxyy}$.

Setting $\Phi_y = -\frac{1}{g}\Phi_{tt}$ (from equation (5)) and $\Phi_{yy} = -\Phi_{xx}$ (from equation (4)), we obtain

$$(17)\qquad \Phi_{tt} = gh\,\Phi_{xx} + \frac{h^2}{2}\Phi_{xxtt} - \frac{gh^3}{6}\Phi_{xxxx} \quad ,$$

Now we want to set $\Phi(x,t) = \phi e^{i(\sigma t+mx)}$, and so we get:

$$(18)\qquad -\sigma^2 = -ghm^2 + \frac{\sigma^2 m^2 h^2}{2} - \frac{gh^3}{6}m^4$$

or

$$+\sigma^2\left(\frac{m^2h^2}{2} + 1\right) = ghm^2\left(1 + \frac{h^2}{6}m^2\right)$$

$$\sigma^2 = ghm^2\left(1 + \frac{h^2}{6}m^2\right)\left(\frac{m^2h^2}{2} + 1\right)^{-1}$$

$$(19)\qquad \sigma^2 = ghm^2\left(1 - \frac{1}{3}h^2m^2 - \dots\right) \quad .$$

Equation (19) corresponds to the exact relation $\sigma^2 = mg\tanh mh$, and we see therefore that (19) is correct up to terms of order h^3.

With the above-obtained results we want now to set up the equations for calculating the ratio of amplitudes of the reflected wave to the incident wave with different boundary conditions.

§1.2.

Suppose an obstacle of the length 2a floats on the surface of the water, which extends from $-\infty$ to $+\infty$ along the x-axis, as shown in Figure 1.

We consider three regions, as follows:

1st region: $-\infty \le x \le -a$ in this region , $\phi \equiv \phi_1$

2nd region: $+a \le x \ge -a$ in this region , $\phi \equiv \phi_3$

3rd region: $+a \le x \le +\infty$ in this region , $\phi \equiv \phi_2$.

We write ϕ_1 and ϕ_2 in the form:

$$(20)\qquad \phi_1 = Be^{imx} + B^*e^{-imx}$$

$$(21)\qquad \phi_2 = Te^{imx} \quad ,$$

corresponding to an incident wave of amplitude B, a reflected wave of amplitude B^*, and a transmitted wave of amplitude T.

For ϕ_3, we have

$$(22) \qquad \phi_3 = \frac{i\sigma w}{2h} x^2 + \gamma x + \delta$$

(see equation (13). From the conservation of energy law we get that ϕ, ϕ' must be continuous at $x = -a$ and $x = +a$. This leads to:

$$(i) \qquad Be^{ima} + B^* e^{ima} = - \frac{i\sigma wa^2}{2h} - \gamma a + \delta$$

$$(ii) \qquad im(Be^{ima} + B^* e^{ima}) = \frac{i\sigma wa}{2h} - \gamma$$

$$(iii) \qquad Te^{ima} = \frac{i\sigma wa^2}{2h} + \gamma a + \delta$$

$$(iv) \qquad imTe^{ima} = - \frac{i\sigma wa}{h} + \gamma \quad .$$

Now we determine w (horizontal displacement) from Newton's law

$$(23) \qquad M \frac{d^2\eta}{dt^2} = -\rho \, \Phi_t - \rho \, g \eta \quad .$$

From equations (23) and (13) we obtain

$$(24) \qquad \int_{-a}^{a} \left[- \frac{i\sigma}{g} \left(- \frac{i\sigma wx^2}{2h} + \gamma x + \delta \right) - w \right] dx = - \frac{\sigma^2 w}{\rho g} M \quad ,$$

and the equation for w is

$$(v) \qquad \alpha w - \frac{2i\sigma a}{g} \delta = 0$$

where

$$\alpha = \frac{\sigma^2 a^3}{3gh} + 2a - \frac{\sigma^2 M}{\rho g} \quad .$$

If we want to take into account the angle of rotation around the center of gravity (for small angles ω) we have the additional equation (for derivation see J. J. Stoker [4])

$$(vi) \qquad \omega = \frac{2\rho i\sigma a^3 \gamma}{3\left(I\sigma^2 - \frac{a^5}{15}\frac{\sigma^2 \rho}{h} - \frac{2}{3}\rho g a^3\right)} \quad .$$

We consider four different cases.

1st case: The obstacle is fixed ($w = 0$). We have to solve only four equations. The result is:

$$\left|\frac{B^*}{B}\right| = \frac{am}{\sqrt{a^2m^2+1}} \qquad \text{(see Figure 2)} \quad . \tag{25}$$

2nd case: The motion of the obstacle is prescribed ($B = 0$),

$$\left|\frac{B^*}{w}\right| = \frac{a^2\sigma}{h}\,\frac{1}{\sqrt{a^2m^2+1}} \quad . \tag{26}$$

For large am we get:

$$|B^*| = \frac{\sqrt{g}}{\sqrt{h}}\,aw \quad .$$

Figure 3 gives $|B^*/w|$ against $2a/\lambda$ and we see that the curves are straight lines with slopes depending on the water depth and the wave length.

In the 3rd case we consider a freely floating obstacle. The obstacle moves horizontally due to the incident wave. The result is as follows:

$$\left|\frac{B^*}{B}\right| = \frac{\frac{1}{2}m^2a^2 + 1}{\sqrt{im^2\left(\frac{3}{2}a + \frac{1}{2}\alpha\frac{gh}{\sigma^2a^2}\right) + \left[m^2\left(\frac{a^2}{2} + \frac{\alpha\, gh}{2\sigma^2a}\right) + 1\right]^2}} \quad .$$

Substituting the value of $\sigma^2 = ghm^2$ we get:

$$\left|\frac{B^*}{B}\right| = \frac{\frac{1}{2}m^2a^2 + 1}{\sqrt{\left(\frac{5}{3}am + \frac{1}{ma} - \frac{1}{2}\frac{Mmh}{a^2\rho}\right)^2 + \left(\frac{2}{3}a^2m^2 + 2 - \frac{hMm^2}{2\rho a}\right)^2}} \quad . \tag{27}$$

We see that the factor which contains M can be for all practical purposes neglected in comparison to the other terms because by assumption mh is very small ($mh \ll 1$) ρ for water = 1.9375. So we get

$$(28) \qquad \left|\frac{B^*}{B}\right| = \frac{\frac{1}{2}\, m^2a^2 + 1}{\sqrt{\frac{4}{9}\, a^4m^4 + \frac{49}{9}\, a^2m^2 + \frac{1}{a^2m^2} + \frac{22}{3}}} \quad .$$

As the equation (28) shows, $|B^*/B|$ is independent of $\underline{h}$.

For the final case, 4, we want to write down the formula for $|B^*/B|$ for an obstacle which is freely floating, while taking into account the rotation around the center of gravity of the obstacle. (This is understood always.)

$$(29) \qquad \left|\frac{B^*}{B}\right| =$$

$$= \frac{\sqrt{\left[-1 + \frac{m^2a^2}{3} - \frac{m^2h\alpha}{3} - \frac{\beta h}{a^2}(m^2a^2 - 2m^2\alpha h - 2)\right]^2 + \left(\frac{mh}{2a}\,\beta\right)^2}}{\sqrt{\left[1 + \frac{m^2a^2}{2} - \frac{\beta h}{a^2}(m^2a^2 - 2m^2\alpha h + 2) + \frac{m^3h\alpha}{3}\right]^2 + \left[m\left(\alpha\,\frac{h}{2a} + 3\beta\,\frac{h}{a}\right) + \beta^2\,\frac{\alpha h^2}{a^3} + \frac{5}{6}\,a\right]^2}}$$

where $\alpha = \frac{M}{2\rho a} - \frac{a^2}{6h} - \frac{g}{\sigma^2}$, $\beta = \frac{3}{2}\left[\frac{M}{\rho a} - \frac{a^2}{15h} - \frac{2}{3}\,\frac{g}{\sigma^2}\right]$.

If we neglect the terms which contain the mass (for the same reason as above) we find:

$$(30) \qquad \left|\frac{B^*}{B}\right| = \sqrt{\frac{\left(\frac{37}{90}\, a^2m^2\right)^2 + \left(\frac{1}{20}\, am + \frac{1}{2am}\right)^2}{\left(2.4 + \frac{52}{90}\, a^2m^2 + \frac{4}{a^2m^2}\right)^2 + \left[\frac{29}{60}\, am + \frac{89}{30am} + \frac{2}{a^2m^3}\right]^2}} \quad .$$

If we compare equation (30) with equation (28) we can see that in equation (30) the value for $|B^*/B|$ is smaller than in equation (28). For large am the difference is very small, (about 2-3%).

§2

Here we want to consider the same problem as in Section 1.2, Part I, but taking into account the terms of the order $O(h^2)$ and $O(h^3)$. For one special case we want to compare the results obtained in the previous section with the result obtained by taking into account the terms of $O(h^2)$ and $O(h^3)$. We have three regions, as before:

1st region: $-\infty \leq x \leq -a$; $\phi \equiv \phi_1$

2nd region: $+a \leq x \geq -a$; $\phi \equiv \phi_3$

3rd region: $+a \leq x \leq +\infty$; $\phi \equiv \phi_2$.

In the previous section we obtained (from equations (17) and (7))

$$(31) \quad g\eta_t = gh\,\Phi_{xx} + \frac{h^2}{2}\Phi_{xxtt} - \frac{gh^3}{6}\Phi_{xxxx}$$

writing for $\eta(x) = w(x)e^{i\sigma t}$; $\Phi(x,t) = \phi(x)e^{i\sigma t}$ and multiplying both sides by $6/gh^3$, we obtain

$$(32) \quad \frac{i6\sigma}{h^3}w = \phi_{xx}\left(\frac{6}{h^2} - \frac{3\sigma^2}{gh}\right) - \phi_{xxxx} \quad .$$

Let us consider for simplicity the case $w = 0$ (no horizontal motion of the obstacle, as in case 1 of the previous section). The solution for this case for the differential equation (32) is:

$$(33) \quad \phi = c_1\cos\mu x + c_2\sin\mu x + c_3 x + c_4$$

where

$$(34) \quad \mu = +\sqrt{\frac{6}{h^2} - \frac{3\sigma^2}{gh}} \quad .$$

Putting in the value for σ from equation (19) we get:

$$(35) \quad \mu = \frac{1}{h}\sqrt{6 - 3m^2h^2 + m^4h^4} \quad .$$

Now we assume, as before,

$$(36) \quad \phi_1 = Be^{imx} + B^*e^{-imx}$$

$$(37) \qquad \phi_2 = Te^{imx}$$

$$(38) \qquad \phi_3 = c_1 \cos \mu x + c_2 \sin \mu x + c_3 x + c_4 \quad .$$

The boundary conditions which are determined from the energy law give us ϕ_3, ϕ_3', ϕ_3''', continuous at $x = \pm a$. Having six conditions and six unknowns $(c_1, c_2, c_3, c_4, T, B^*)$, we can solve for $|B^*/B|$ and we get:

$$(39) \qquad \left|\frac{B^*}{B}\right| =$$

$$= \frac{(-a\mu^4 + 2am^2\mu^2 - am^4)\cos \mu a + \left(\frac{m^4}{\mu} - m^2\mu\right)\sin \mu a}{\sqrt{\left[(a\mu^4 - 2am^2\mu^2 + am^4)\cos \mu a + \left(m^2\mu - \frac{m^4}{\mu}\right)\sin \mu a\right]^2 + \left[\mu^2\left(\frac{\mu^2}{m} - m\right)\cos \mu a\right]^2}} \quad .$$

If we compare equation (39) with equation (25), it is easy to verify that for small h/λ $(h/\lambda < .1)$ and $a/\lambda \geq 1$ $(m = 2\pi/\lambda)$, $|B^*/B|$ will have this same value. For $a/\lambda < 1$ equation (39) shows that the equation (25) may not yield accurate results.

PART II

§1.

We have a water surface of depth h. At a certain interval, from -a to +a, a force $p = \delta(x)e^{i\sigma t}$ is applied. We ask for the potential function at $x = \infty$.

We consider only irrotational incompressible flow. We restrict ourselves to two-dimensional motion in an x,y-plane. Under these assumptions we have

$$(1) \quad \Phi_{xx} + \Phi_{yy} = 0 \qquad y < 0 \quad ; \quad t > 0$$

$$(2) \quad \Phi_y = 0 \qquad \text{at} \quad y = -h$$

$$(3) \quad g\,\Phi_y + \Phi_{tt} = -\frac{i\delta}{\rho}\left(\delta(x)e^{i\sigma t}\right) = -\int_{-a}^{a} \frac{i\sigma}{\rho}\,\delta(x-\xi)e^{i\sigma t}d\xi$$

$$\text{for} \quad y = 0 \quad , \quad t > 0$$

$$(4) \quad p(x,t) = \delta(x)e^{i\sigma t} \qquad \text{for} \quad t > 0 \quad .$$

In other words, the equation (4) means that an oscillatory pressure is applied at the origin on the free surface and maintained there for all time.

Now we have to find a solution $\Phi(x,y,t)$ of (1) which behaves properly at ∞, and which satisfies the free surface and bottom conditions. The result of transforming (1), (see Sneddon [7]),

$$(5) \quad -s^2\,\bar{\Phi} + \bar{\Phi}_{yy} = 0$$

$$(6) \quad \bar{\Phi}(s,y,t) = A(s,t)\cosh s(y+h) \quad .$$

Applying the boundary condition to the cosine transformation, we obtain

$$(7) \quad g\,\bar{\Phi}_y + \bar{\Phi}_{tt} = -\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \frac{i\sigma}{\rho}\,e^{i\sigma t}e^{i\xi s}d\xi \quad .$$

On substitution of $\Phi(s,o,t)$ from (5), we find

$$(8) \quad A_{tt}\cosh sh + gsA \sinh sh = \frac{-1}{\sqrt{2\pi}} \frac{i\sigma}{\rho} \int_{-a}^{a} e^{i\sigma t} e^{i\xi s} d\xi \quad .$$

Now the initial conditions are:

$$(9) \quad A(s,o) = A_t(s,o) = 0 \quad ,$$

and the solution of (8) subject to the initial condition (9) is

(10)

$$A(s,t) = \frac{-1}{\sqrt{2\pi}} \frac{i\sigma}{\rho} \int_0^t \frac{e^{i\sigma(t-\tau)} \sin\sqrt{gs \tanh s(y+h)}\,\tau}{\cosh s(y+h)\sqrt{gs \tanh s(y+h)}} d\tau \int_{-a}^{a} e^{i\xi s} d\xi \ .$$

Finally, we insert the last expression for $A(s,t)$ in (6) and, applying the inverse transformation, we obtain

(11)

$$\Phi(x,y,t) = \frac{-i\sigma}{\rho\pi} \int_0^{\infty} \cos sx \int_0^t \frac{e^{i\sigma t} \sin\sqrt{gs \tanh sh}\,\tau}{\sqrt{gs \tanh sh}} d\tau \int_{-a}^{a} e^{i\xi s} d\xi dx \ .$$

Now we want to solve the integral $\int_0^t$ for $t \to \infty$.

$$\int_0^t = \frac{e^{i\sigma t}}{\sqrt{gs \tanh sh}} \int_0^t e^{-i\sigma\tau} \sin\sqrt{gs \tanh sh}\,\tau \, d\tau =$$

$$= \frac{e^{i\sigma t}}{\sqrt{gs \tanh sh}} \left[e^{-i\sigma t} \frac{(-i\sigma \sin\sqrt{gs \tanh sh}\,t - \sqrt{gs \tanh sh} \cos\sqrt{gs \tanh sh}\,t)}{gs \tanh sh - \sigma^2} + \frac{\sqrt{gs \tanh sh}}{gs \tanh sh - \sigma^2} \right] \ .$$

Setting for sine and cosine the exponential functions, we get

$$\int_0^t = e^{i\sigma t}\left| \begin{array}{l} -\dfrac{1}{2\sqrt{gs\tanh sh}}\,\dfrac{e^{i(\sqrt{gs\tanh sh}\,-\sigma)t}}{(\sqrt{gs\tanh sh}\,-\sigma)} \\ -\dfrac{1}{2\sqrt{gs\tanh sh}}\,\dfrac{e^{-i(\sqrt{gs\tanh sh}\,+\sigma)t}}{(\sqrt{gs\tanh sh}\,+\sigma)} \\ +\dfrac{1}{gs\tanh sh-\sigma^2} \end{array}\right| \begin{array}{l} (i) \\ (ii) \\ (iii) \end{array} .$$

The parts (i) and (iii) have a singularity at

$$\sigma^2 = gs\tanh s(y+h) \quad .$$

For $t \to \infty$ we can verify that (i), (ii) die out and we have, (see Stoker [4]),

$$(12) \qquad \Phi \approx -\frac{i\sigma}{\rho\pi}\, e^{i\sigma t}\int_{-a}^{a} e^{i\xi s}\int_h \frac{\cos xs}{gs\tanh sh-\sigma^2}\, d\xi\, ds \quad .$$

We integrate over the path $\underline{\underline{h}}$. Or,

$$(13) \qquad \Phi \approx +\frac{\sigma}{2\rho\pi i}\, e^{i\sigma t}\int_h \frac{e^{-isx}(e^{ias}-e^{-ias})+e^{isx}(e^{ias}-e^{-ias})}{s(gs\tanh sh-\sigma^2)}\, ds \; .$$

e^{isx} makes a contribution that tends to zero for $x \to \infty$. The second part of the integral gives

$$\frac{1}{2\pi i}\int_h \frac{e^{-isx}(e^{ias}-e^{-ias})}{gs\tanh sh-\sigma^2}\, ds \simeq \frac{e^{-is_o x}(e^{ias_o}-e^{-ias_o})}{s_o(g\tanh s_oh+gsh-gs_oh\tanh^2 sh)} \; .$$

Or, inserting into (13),

$$(14) \qquad \Phi \approx -\frac{2\sigma}{\rho g}\, e^{i(\sigma t-s_o x)}\, \frac{\sin as_o}{s_o\tanh s_oh+s_o^2h-s_o^2h\tanh^2 s_oh} \; .$$

For s_oh very small (where $s_o = 2\pi/\lambda$, λ = wave length, h =

depth of the water) we get:

$$(15) \qquad \Phi \approx -\frac{\sigma}{\rho g} e^{i(\sigma t - s_o x)} \frac{\sin a s_o}{s_o^2 h} \quad .$$

§2. The Potential Function Φ in the Approximate Theory (Shallow Water Theory).

From Newton's law (see Part I, equation (23)) we have

$$(16) \qquad p = -\rho \, \Phi_t - \rho g \eta$$

or

$$(17) \qquad -\eta_t = \frac{p_t}{\rho g} + \Phi_{tt} \frac{1}{g} \quad .$$

From equation (11), Part I, we have

$$(18) \qquad \eta_t = -h \, \Phi_{xx} \quad .$$

Setting $\eta = \omega(x)e^{i\sigma t}$, $\Phi = \phi(x)e^{i\sigma t}$, $p = \bar{p}(x)e^{i\sigma t}$, we get:

$$(19) \qquad h\phi_{xx} = \frac{i\sigma\bar{p}}{\rho g} - \frac{\sigma^2}{g} \phi$$

or

$$(20) \qquad \phi + \frac{hg}{\sigma^2} \phi_{xx} + \frac{1}{i\sigma\rho} \bar{P} = 0 \qquad \text{for} \quad -a \le x \le a$$

$$\sigma = \sqrt{gh} k = \frac{2\pi}{\lambda} \sqrt{gh} \quad \text{(from equation (15a), Part I)} .$$

The solution for the differential equation (20) is:

$$(21) \qquad \phi = b \cos \frac{\sigma}{\sqrt{gh}} x - \frac{\bar{P}}{i\rho\sigma} \quad .$$

At $x = +\infty$ we have the transmitted wave which we write in the form

$$(22) \qquad \bar{\phi} = Te^{-ikx} \qquad \text{(for} \quad x = +\infty) \quad .$$

We impose the same boundary condition as in Part I; namely, ϕ, ϕ' have to be continuous at $x = +a$, and so we get:

(23) $$b \cos ka - \frac{\overline{P}}{i\rho\sigma} = Te^{-ika} \qquad \left(k = \frac{\sigma}{\sqrt{gh}}\right)$$

(24) $$\frac{b}{i} \sin ka = Te^{-ika}$$

or

(25) $$T = -\frac{\overline{P}}{\sigma\rho} \sin ka \quad ,$$

and now $\overline{\phi}$;

(26) $$\overline{\phi}e^{i\sigma t} = -Te^{-ikx}e^{i\sigma t} = -\frac{\overline{P}}{\sigma\rho} \sin ka \; e^{i(\sigma t-kx)} = \underline{\Phi}$$

or:

(27) $$\underline{\Phi} = -e^{i(\sigma t-kx)} \sin ka \; X \frac{\sigma}{\rho g h k^2} \qquad (\text{for} \quad p = 1) \quad .$$

We see that for small kh equation (15) which is obtained from the exact theory is the same as equation (27) of the approximate theory (shallow water theory). For large kh (k given) the ratio of $\underline{\Phi}$ shallow ($\underline{\Phi}$ obtained from the approximate theory) to $\underline{\Phi}_p$ ($\underline{\Phi}$ obtained by the exact theory) is given by

(28) $$\left|\frac{\underline{\Phi}_{sh}}{\underline{\Phi}_p}\right| = \frac{1}{2\sqrt{kh}} \quad .$$

In this case we have also a phase difference because σ is not equal in both cases. Figure 5 shows the ratios of $|\underline{\Phi}_{sh}/\underline{\Phi}_p|$ for given h/λ.

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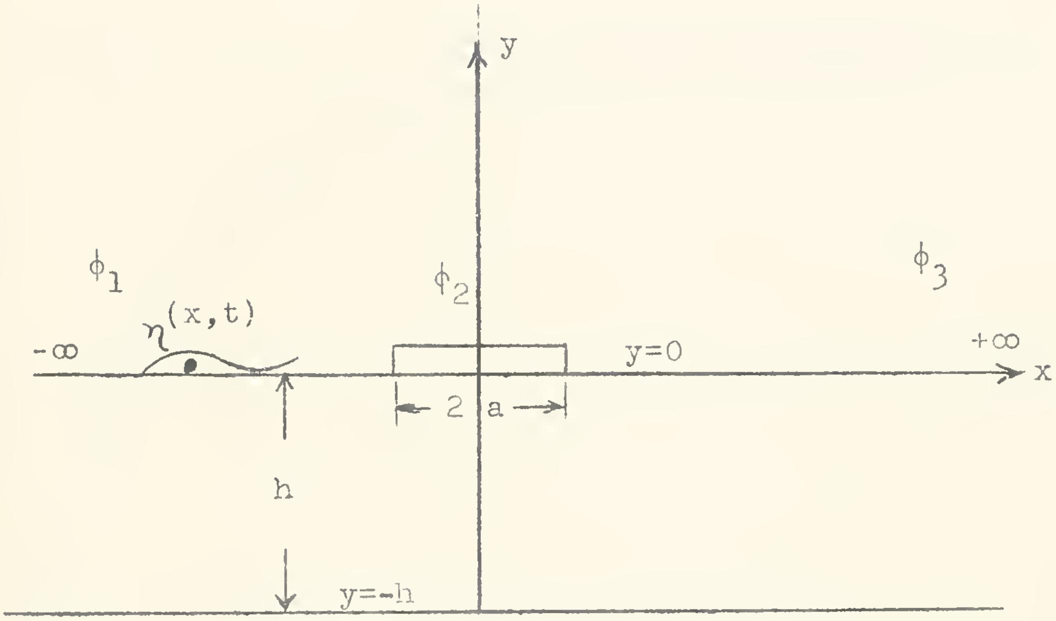
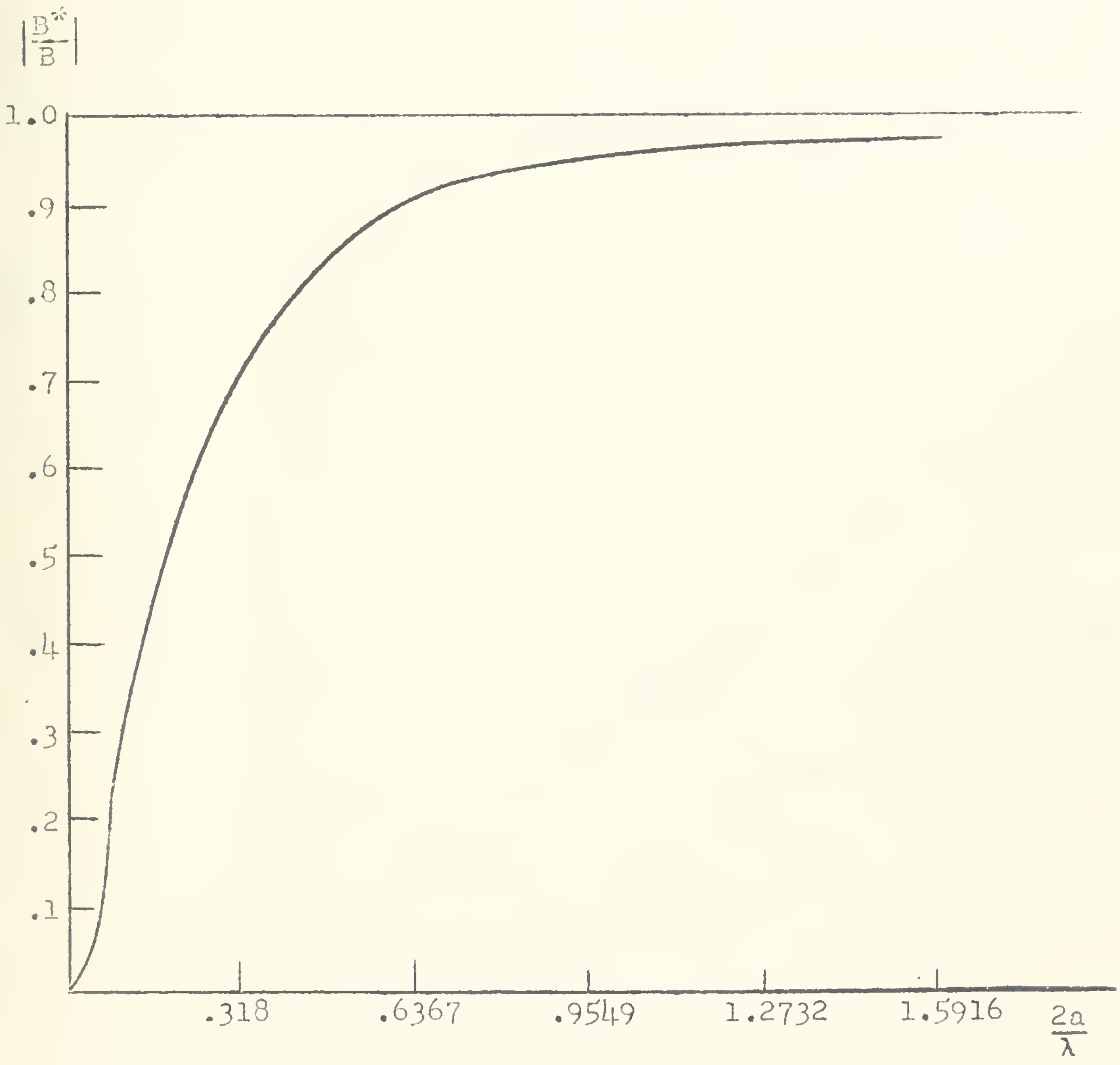
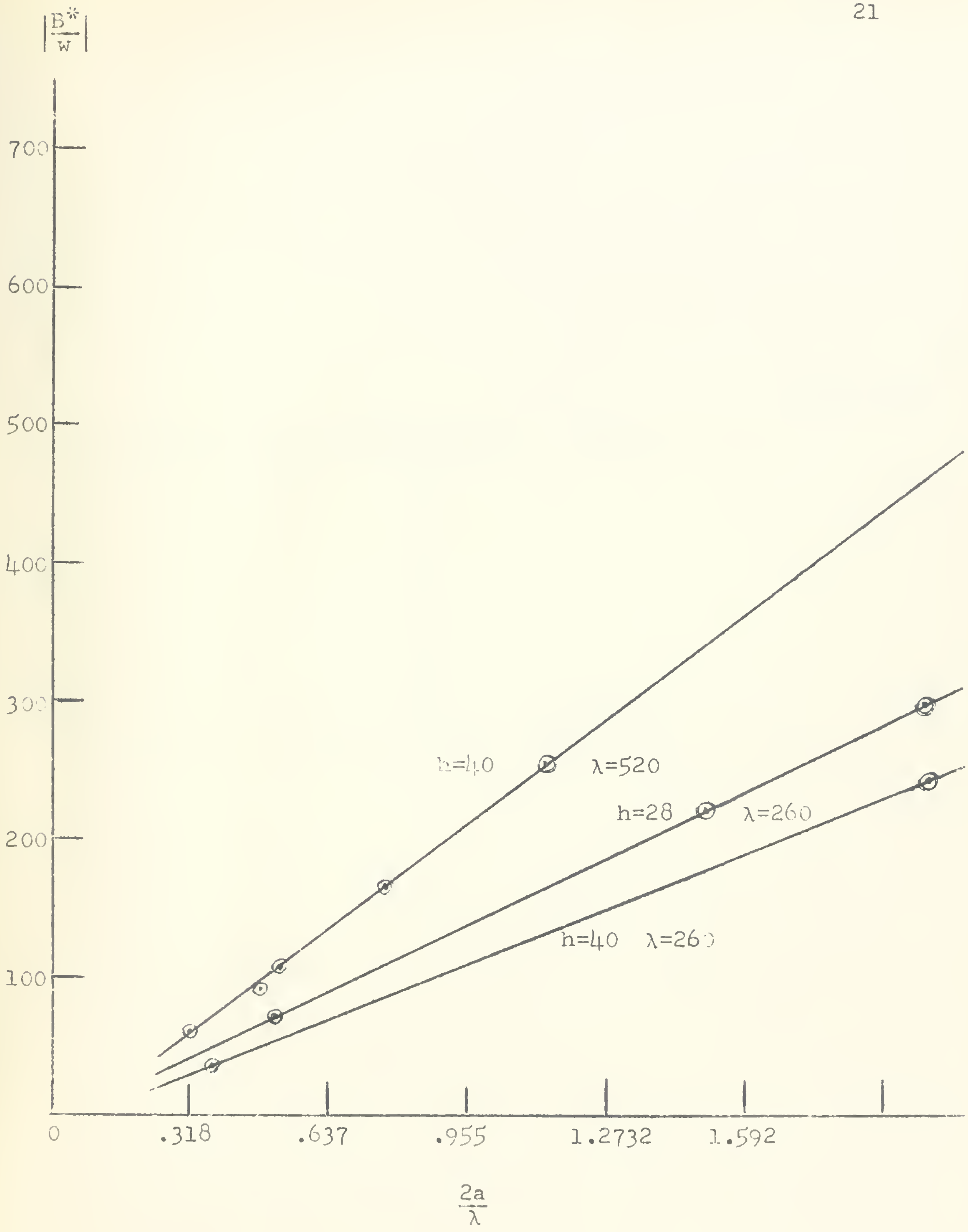


Figure 1



$\left|\frac{B^*}{B}\right|$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.318
.6367
.9549
1.2732
1.5916
$\frac{2a}{\lambda}$

Figure 2



$\left|\frac{B^*}{w}\right|$
700
600
500
400
300
200
100
0
.318
.637
.955
1.2732
1.592
$\frac{2a}{\lambda}$
h=40 λ=520
h=28 λ=260
h=40 λ=260

Figure 3

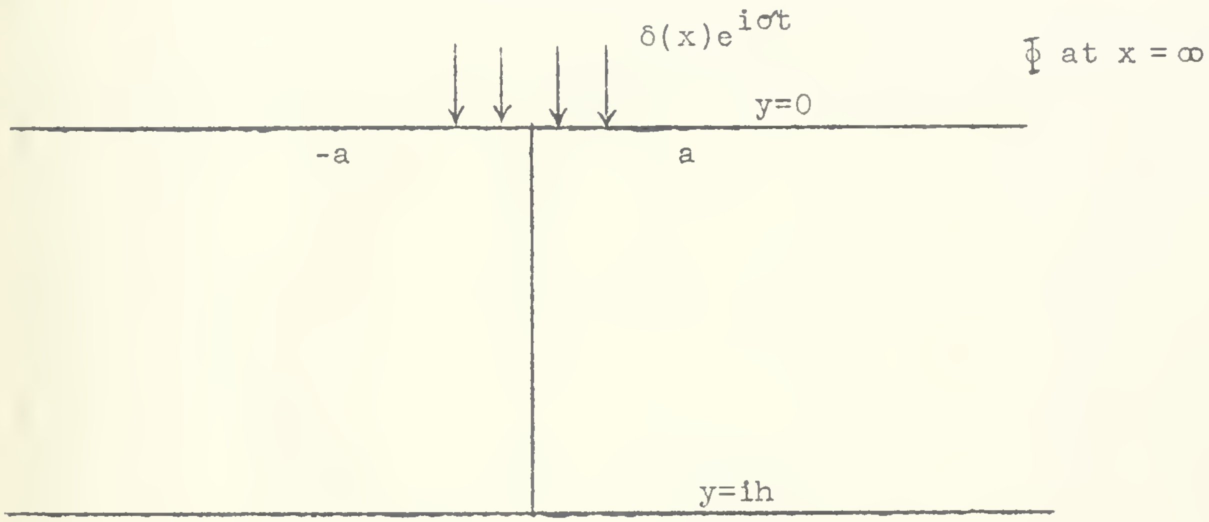


Figure 4

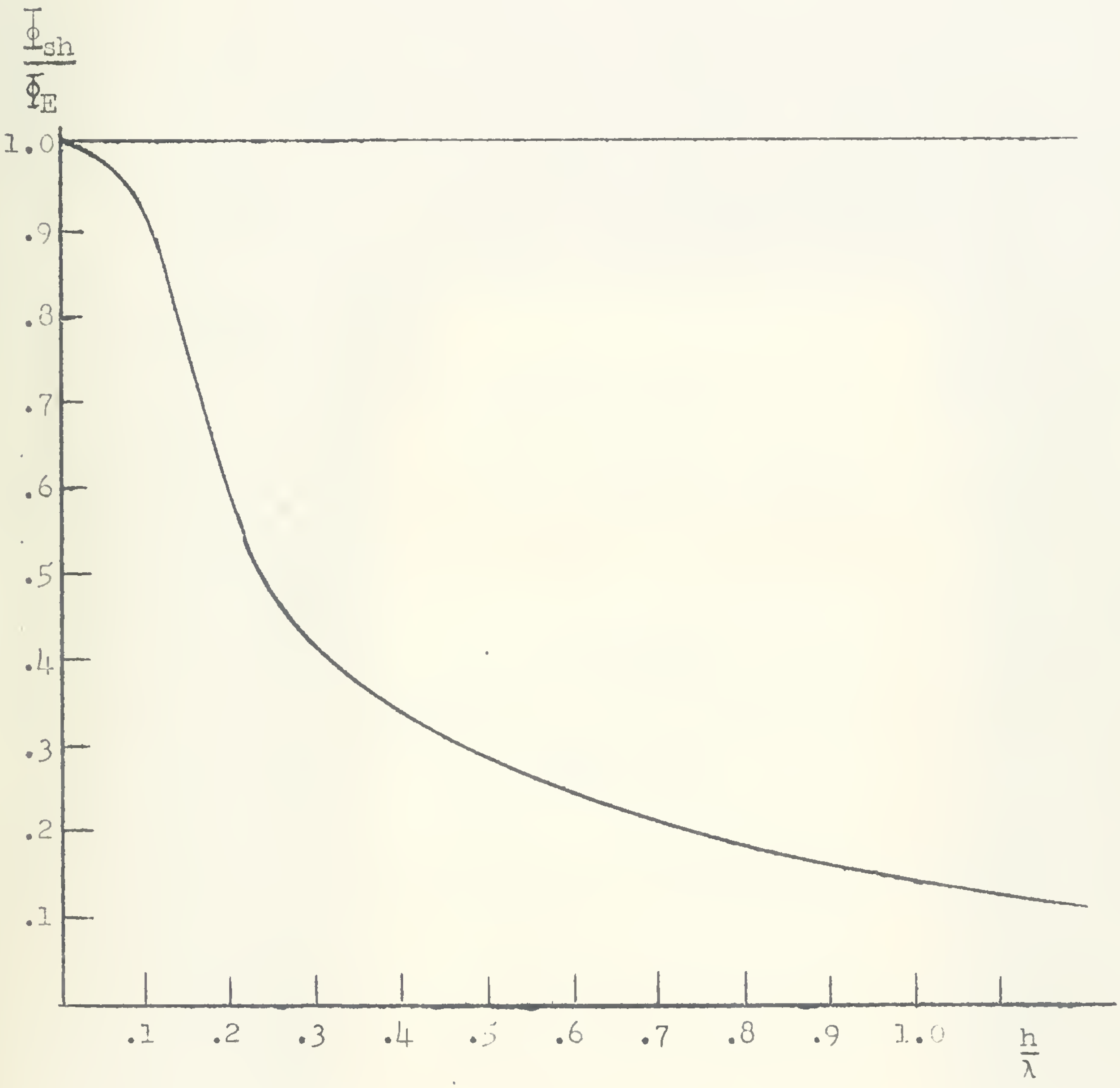


Figure 5

Date Due

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